

Iterative learning extremum seeking tracking via Lie bracket approximation

Zhixing Cao, Hans-Bernd Dürr, Christian Ebenbauer, Frank Allgöwer, Furong Gao

Abstract—In this paper, we develop an extremum seeking control method integrated with iterative learning control to track a time-varying optimizer within finite time. The behavior of the extremum seeking system is analyzed via an approximating system – the modified Lie bracket system. The modified Lie bracket system is essentially an online integral-type iterative learning control law. The paper contributes to two fields, namely, iterative learning control and extremum seeking. First, an online integral type iterative learning control with a forgetting factor is proposed. Its convergence is analyzed via k -dependent (iteration-dependent) contraction mapping in a Banach space equipped with λ -norm. Second, the iterative learning extremum seeking system can be regarded as an iterative learning control with “control input disturbance”. The tracking error of its modified Lie bracket system can be shown uniformly bounded in terms of iterations by selecting a sufficiently large frequency. Furthermore, it is shown that the tracking error will finally converge to a set, which is a λ -norm ball. Its center is the same with the limit solution of its corresponding “disturbance-free” system (the iterative learning control law); and its radius can be controlled by the frequency.

Index Terms—Contraction mapping, extremum seeking (ES), iterative learning control (ILC), Lie bracket, λ -norm

I. INTRODUCTION

CONSIDERABLE research efforts have been devoted to extremum seeking control for the last several decades. The mechanism of extremum seeking control is to optimize a certain system performance measure (cost function) by adaptively adjusting the system parameters merely according to output measurements of the plant. Since there is little knowledge required about the plant dynamics, extremum seeking control has attracted the attention from various engineering domains, e.g., bioreactor, combustion, compressor [1]–[3]. In most of the classic extremum seeking literatures, the basic assumption is that the static input-output mapping is *time invariant*, i.e., [2], [4], [5] and the references therein. However, time varying input-output mapping is common in practice. For example, injection molding engineers are quite interested in finding an optimal ram velocity profile to minimize the unevenness of polymer melt front velocity (PMFV) in the mold cavity to yield a smooth surface on the final product. The map from ram velocity to PMFV is typically time varying, due to the complex geometry in the cavity [6].

Zhixing Cao and Furong Gao are with the Department of Chemical and Biomolecular Engineering, Hong Kong University of Science and Technology, Hong Kong SAR. Hans-Bernd Dürr, Christian Ebenbauer and Frank Allgöwer are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, 70569 Germany. E-mail: email: {zcaob, kefgao}@ust.hk, {hans-bernd.duerr, ce, frank.allgower}@ist.uni-stuttgart.de.

In this paper, we propose a novel extremum seeking scheme based on iterative learning control to find the optimizer (optimal trajectory) of a time varying mapping. Particularly, the contributions of the paper are threefold and summarized as follows.

First, a novel extremum seeking approach is developed to solve the optimizer tracking problem by introducing an analog memory into the extremum seeking loop, which differs from the existing methods [1], [7]–[11]. There are two categories of similar approaches concerning the time varying mappings reported in literatures. Wang and Krstić introduced a detector to minimize the amplitude of stable limit cycle by tuning a controller parameter to a constant optimizer [7]. Guay and his colleagues employed system flatness to parameterize all the variables by sine and cosine series; extremum seeking was used to steer the coefficients of the series to the optimizer [8]. Haring and his coworkers have developed a mean-over-perturbation-period filter to produce an estimate of the gradient for extremum seeking loop [9]. The underlying assumption of all these methods is that the corresponding cost functions, although time varying, admit a constant optimizer, which is different from our discussion in this paper. As for the second type, Krstić introduced a compensator for time varying mapping which is structured by Wiener-Hammerstein models. This method requires the knowledge about the two time varying blocks, which may restrict its applicability [1], [10]. An adaptive delay-based estimator was introduced to feed gradient estimates to extremum seeking loop by Sahneh and his coworkers. The extremum seeking loop is a cascade feedback in nature [11]. Basically, this type of methods employed a fast decay ratio to suppress the tracking error. However, in some circumstances, these methods may not be able to yield satisfying transient tracking performance. Fortunately, many time varying systems exhibit certain repetitive behaviors, such as semiconductor manufacturing, pharmaceutical producing and the injection molding mentioned above [12]. Exploiting such repetitiveness provides potential for circumventing the aforementioned drawbacks. Actually, the works in [8], [9] have already utilized such a feature but not that sufficiently, since they only used it to reduce the cost function into a starting-time-independent function. Iterative learning control, first proposed by Arimoto et. al. [13], is good at exploiting repetitiveness to improve tracking performance from iteration to iteration [12], [14]–[16].

Second, the proposed approach is a new ILC scheme. The fundamental assumption adopted in most ILC literature is that the tracking trajectory has been already available as priori knowledge [17], which renders the controller knowing the

direction to steer. Within this paper, we do not have such an assumption; only the distance to the tracking trajectory is known, i.e., the absolute value of tracking error. Intuitively, almost all the ILC approaches will fail for this situation, since somehow the negative feedback along the iterations cannot be ensured. It is natural to come up with combining extremum seeking with ILC; let them collaborate with each other: ILC provides the past learning experience to extremum seeking to improve transient tracking performance; the “direction” information needed by ILC is given by extremum seeking.

Third, we propose a new online infinite-dimensional integral-type (I-type) ILC control law. The dynamical behavior of the ILC system is studied by k -dependent contraction mapping. The convergence of the ILC system is discussed. Similar results are [18]–[22]. [18] has studied the proportional-type (P-type) online ILC, but only derived an index bound regarding the ultimate tracking performance. Wang has considered the sampling effect and input saturation issues in the offline P-type ILC, and implemented it experimentally [19]. In [20], Saab has investigated the offline P-type and D-type (derivative-type) ILC for the stochastic scenario, where a dynamic learning gain was adopted. [21] has discussed the forgetting factor selection for a general offline ILC algorithm. Ouyang and his colleagues have developed an online PD-type ILC for a class of input-affine nonlinear system, and also presented an ultimate bound of tracking performance [22]. According to authors’ knowledge, there is few papers contributing to online I-type ILC. Furthermore, we present more than an ultimate bound of tracking performance; the limit solution and its uniqueness are studied as well. More interestingly, the nature of the proposed extremum seeking system has been revealed, when the behaviors of the proposed methods are analyzed via an approximating system – *modified Lie bracket system*. It is an online integral-type ILC with “control input disturbance”. Based on the same reasoning, we extend the results on ILC to analyze the iterative learning extremum seeking system. We show that the system is uniformly bounded in terms of k and converges to a set as k goes to infinity. The particular set is a λ -norm ball, whose center is the limit solution of the associated ILC system, and radius can be controlled by the frequency of the sinusoid signal.

The rest of the paper is structured as follows: Section II gives the technical preliminaries about λ -norm and Lie bracket system; Section III formulates the problem; Section IV presents the main results; Section V provides an illustrative example for the theory; a conclusion is drawn and an outlook is given in Section VI.

Notations: \mathbb{N}_{++} and \mathbb{N}_0 denote the set of positive integers excluding and including zero respectively. \mathbb{Q}_{++} is for the positive rational numbers. \mathbb{M}_n is for all the matrices with dimensions $n \times n$. C^n with $n \in \mathbb{N}_0$ stands for the set of n times continuously differentiable functions and C^∞ for the set of smooth functions. The gradient of a continuous function $f \in C^1 : \mathbb{R}^n \rightarrow \mathbb{R}$ is $\nabla_x f(x) \triangleq \left[\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right]^T$. Two vector fields $f, g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ are twice continuously differentiable; their Lie bracket is defined as $[f, g](x, t) \triangleq \frac{\partial g(x, t)}{\partial x} f(x, t) - \frac{\partial f(x, t)}{\partial x} g(x, t)$. For a point $x \in \mathcal{X}$ and a set

$\mathcal{S} \subset \mathcal{X}, x \notin \mathcal{S}$, the distance from x to \mathcal{S} is defined as $\text{dist}(x, \mathcal{S}) = \min_{s \in \mathcal{S}} \|x - s\|$. For two compact sets \mathcal{X}, \mathcal{Y} and $\mathcal{X} \subset \mathcal{Y}$, $\partial \mathcal{Y}$ for the boundary of \mathcal{Y} , the distance is defined as $\text{dist}(\mathcal{X}, \mathcal{Y}) = \min_{x \in \mathcal{X}, y \in \partial \mathcal{Y}} \|x - y\|$. $\text{int } \mathcal{X}$ means the interior of set \mathcal{X} .

II. PRELIMINARIES

A. λ -norm

The λ -norm, introduced by Arimoto et. al. in 1984 [13], is a topological measure widely used to analyze the convergence of ILC control law [23]. The formal definition of λ -norm is as follows.

Definition 2.1: [23] The λ -norm of a function $f : [0, L] \rightarrow \mathbb{R}^n$ is

$$\|f(\bullet)\|_\lambda = \max_{t \in [0, L]} e^{-\lambda t} \|f(t)\|_\infty$$

where $\|f(t)\|_\infty = \max_{1 \leq i \leq n} |f_i(t)|$.

From the definition, it is easy to see that

$$\|f(\bullet)\|_\lambda \leq \|f(\bullet)\|_C \leq e^{\lambda L} \|f(\bullet)\|_\lambda$$

for positive λ , where $\|f(\bullet)\|_C = \max_{t \in [0, L]} \|f(t)\|_\infty$. This shows that the λ -norm is equivalent to the C-norm, which means the convergence with respect to λ -norm is still valid with respect to C-norm. Its advantage is that a non-monotonically converged sequence on C-norm can be monotonically converged on λ -norm for a properly chosen λ .

B. Lie bracket system

In the classic extremum seeking literatures, for example [4], the behavior of the original extremum seeking system is analyzed by averaging. However, within this paper, an emerging analysis tool based on the Lie bracket approximation is going to be used to study the extremum seeking system. For an input-affine extremum seeking system

$$\dot{x} = b_1(x) \sqrt{\omega} u_1(\omega t) + b_2(x) \sqrt{\omega} u_2(\omega t)$$

with $\omega \in (0, \infty)$, its Lie bracket system is

$$\dot{z} = \frac{1}{2} [b_1, b_2](z)$$

For instance, in a traditional ES system, $b_1(x) = 1, b_2(x) = -\alpha f(x)$, $\alpha > 0, f(x) \in C^2 : \mathbb{R}^n \rightarrow \mathbb{R}$ admitting a local minimum, its Lie bracket system is $\dot{z} = -\alpha \nabla f(z)/2$, which clearly minimizes the cost function. One advantage of using Lie bracket approximation is that the approximating system and the original system share the same rate of convergence, i.e., the exponential stability of the Lie bracket system implies the practical exponential stability of the original one. For details, please refer to [24].

III. PROBLEM STATEMENT

In this paper, we study the following optimization problem: at each time $t \in [0, L]$, we solve the

$$\min_{x(t)} F(x(t), t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ and $F : \mathbb{R}^n \times [0, L] \rightarrow \mathbb{R}$,

$$F(x(t), t) = f^*(t) + [x(t) - x^*(t)]^T Q [x(t) - x^*(t)] \quad (2)$$

$Q \in \mathbb{M}_n$ is a positive definite matrix. $L > 0$ is the finite time duration or period. For each time t , $x(t) = x^*(t)$ is the solution to the optimization problem (1) with the corresponding optimal value $f^*(t)$. For all time $t \in [0, L]$, all the $x^*(t)$ form a time varying optimizer/minimizer (or called optimizer/minimizer trajectory) $x^* : [0, L] \rightarrow \mathbb{R}^n$; if $x^*(t) \equiv c$ for any t , c a constant vector, then x^* is named a constant optimizer. Similarly, the optimal trajectory $f^* : [0, L] \rightarrow \mathbb{R}$ consists of all the optimal value $f^*(t)$ over the interval $[0, L]$. $x(t)$ in (1) or its ensemble x is named optimization variable or simply input. Obviously, the study on minimization problem does not loss any generality; it can be easily extended to the maximization problem by only altering the sign of the extremum seeking gain. According to Ariyur and Krstić (pp. 21) [1], the above-mentioned form of parameterization is able to approximate any vector function $F(x(t), t)$ with a quadratic minimum at $x^*(t)$. The objective for extremum seeking is to steer x to the minimizer trajectory x^* over $[0, L]$.

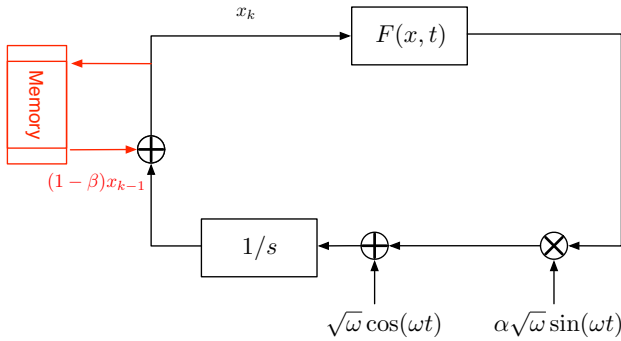


Fig. 1: Block diagram of iterative learning extremum seeking

A. Classical approach

The input x in the conventional ES literatures, is driven to the neighborhood of the minimizer trajectory x^* by solving the following ordinary differential equation (ODE).

$$\dot{x}(t) = -\alpha F(x(t), t) \sqrt{\omega} \sin(\omega t) + \sqrt{\omega} \cos(\omega t) \quad (3)$$

Here $\alpha \in (0, \infty)$ is a constant gain. The black part in Fig. 1 presents the closed loop of this classic approach. For a fast changing minimizer trajectory, this approach may fail to achieve a satisfactory tracking within finite duration.

B. Main idea

Since the repeated operation mode of the repetitive process [12], there is no need to solve the tracking problem in only single iteration; instead, it can be solved in many iterations even infinite iterations. Since the previous input is a good approximation of the optimizer trajectory, it is quite handy and natural to modify the previous input to generate a new

control input. Borrowing the ideas from ILC, we propose to solve the ES tracking problem by solving the following ODE.

$$\dot{x}_k(t) = (1 - \beta) \dot{x}_{k-1}(t) - \alpha F(x_k(t), t) \sqrt{\omega} \sin(\omega t) + \sqrt{\omega} \cos(\omega t) \quad (4)$$

The subscript $k \in \mathbb{N}_{++}$ indicates the iteration index; $\beta \in (0, 1]$ is the forgetting factor, which has been adopted in many ILC literatures, i.e., [18]. To keep the notation simple, x_k will be used in the rest of paper instead of $x_k(t)$ without any ambiguity and the same for other similar variables, unless stated otherwise.

Physically, (4) introduces a memory storing the input (x_{k-1}) of the previous batch into the extremum seeking loop as the red part shown in Fig. 1. The term x_{k-1} is the feed-forward component, while the second and third terms in the right hand side of (4) are the feedback component. It is intuitively understandable that the proposed method could result in a better performance than the conventional one (3), since the feed-forward term somehow can facilitate the tracking. Meanwhile, the mechanism is always feeding a new input into the system by insistently using feedback information to “polish” x_{k-1} , a rough “guess” of the minimizer. Thus, it can be expected that the tracking performance would improve gradually as the rough “guess” is becoming finer.

Remark 3.1: It is noted that when $k = 1$, (4) becomes $\dot{x}_1 = (1 - \beta) \dot{x}_0 - \alpha F(x_1, t) \sqrt{\omega} \sin(\omega t) + \sqrt{\omega} \cos(\omega t)$, which is exactly the standard extremum seeking, if $\dot{x}_0(t)$ and $x_0(t)$ are assumed to be zero, or equivalently the memory is reset to zero.

Since we are only interested in the system over the finite interval, there is no need to discuss its asymptotic stability along the time direction. The problems we are more interested in are under what condition x_k will approach to a small neighborhood of x^* when k tends to infinity, i.e., $\|x_k - x^*\|_C < D, k \rightarrow \infty$ and what determines D .

IV. MAIN RESULTS

Note that if (3) iterates itself to different k , the corresponding coefficients on $\cos(\omega t)$ will be different. On the other hand, the Lie bracket approximation is only valid for systems with respect to t not to both k and t . Thus, we cannot directly employ it but have to tailor it accordingly. We propose to use the *modified Lie bracket (MLB)* system to approximate the original one.

$$\begin{aligned} \dot{z}_k &= (1 - \beta) \dot{z}_{k-1} + \frac{\gamma_k}{2} [1, -\alpha F](z_k, t) \\ &= (1 - \beta) \dot{z}_{k-1} - \frac{\alpha \gamma_k}{2} \nabla_z F(z_k, t) \end{aligned} \quad (5)$$

Here the only modification is the introduction of γ_k . γ_k is a compensating parameter and only related to the forgetting factor β as defined below.

$$\gamma_k = \frac{1 - (1 - \beta)^k}{\beta} \quad (6)$$

It is evident that $\{\gamma_k\}$ is a monotonically increasing sequence and $1 \leq \gamma_k \leq 1/\beta$. The first equality holds if and only if $k = 1$, which implies that the MLB system (5) reduces to the traditional Lie bracket system when $k = 1$. The term \dot{x}_{k-1} is a

feed-forward term and only a function of time with respect to the current iteration k , and does not contain any sinusoid term. Thus, it should not be included into the Lie bracket according to the theory in [24].

Remark 4.1: Observing (5), the MLB system (5) is unlike the conventional Lie bracket system, because of the existence of the derivation of x_{k-1} rather than being an independent system of itself. Inserting (2) into (5) and denoting

$$\Gamma_k = \frac{\alpha\gamma_k Q}{2} = \frac{[1 - (1 - \beta)^k]\alpha Q}{2\beta}$$

we can rewrite (5) as

$$\dot{z}_k = (1 - \beta)\dot{z}_{k-1} - \Gamma_k(z_k - x^*) + (1 - \beta)(\dot{x}_{k-1} - \dot{z}_{k-1}) \quad (7)$$

Before presenting our main results, we impose the following assumptions.

- A1) The time varying optimizer $x^* \in C^1 : [0, L] \rightarrow \mathbb{R}^n$ and optimal trajectory $f^* \in C^1 : [0, L] \rightarrow \mathbb{R}$;
- A2) The initial condition of each iteration of (4) is identical and equal to zero, i.e. $x_k(0) = x(0) = 0, \forall k$; so is the MLB system (5), i.e. $z_k(0) = x(0) = 0$ for all k ;
- A3) Assume that Q is bounded and $Q \geq \delta I$, where δ is a known positive real number.

Remark 4.2: A1 is assumed to ensure the existence of $F(x, t)$'s first-order partial derivative $\frac{\partial F(x, t)}{\partial t}$, which is required by integration by parts. A2 is a common assumption in majority of ILC literatures to simplify derivations [14], called *identical initial condition (i.i.c)*. The second part of A2 is assumed to let MLB system be in accordance with the original system. A3 means that we do not require exact knowledge about Q , the Hessian matrix, but requires a lower bound of Q for the minimum case, since a known Q means having a precise knowledge about the plant, which is generally impossible in practice. δ is only required by the following conceptual analysis, does not restrict our method's applicability.

Taking integration on both sides of (7), it turns to be

$$z_k = (1 - \beta)z_{k-1} - \Gamma_k \int_0^t (z_k - x^*) ds + (1 - \beta)(x_{k-1} - z_{k-1}) \quad (8)$$

The z_k can be interpreted as control input with $z_k - x^*$ as tracking error. Then, the rewriting above clearly shows that it is in the form of integral-type ILC online (feedback) control [14] with some "control input disturbance", i.e., $(1 - \beta)(x_{k-1} - z_{k-1})$. ■

Note that (5) is a linear ordinary differential equation, and we can write down its explicit solution.

$$z_k = e^{-\Gamma_k t} z(0) + \int_0^t e^{-\Gamma_k(t-s)} [\Gamma_k x^* + (1 - \beta)\dot{x}_{k-1}] ds$$

It follows from $z(0) = 0$ and integrating by parts that

$$z_k = (1 - \beta)e^{-\Gamma_k t} \left[e^{\Gamma_k s} x_{k-1} \Big|_0^t - \int_0^t e^{\Gamma_k s} \Gamma_k x_{k-1} ds \right] + e^{-\Gamma_k t} \int_0^t e^{-\Gamma_k s} \Gamma_k x^* ds$$

Because $x_{k-1}(0) = 0$, we have

$$z_k = (1 - \beta)x_{k-1} + e^{-\Gamma_k t} \int_0^t e^{\Gamma_k s} \Gamma_k [x^* - (1 - \beta)x_{k-1}] ds \quad (9)$$

Now we define the tracking error of the MLB system (5) as $y_k \triangleq z_k - x^*$; the error system is

$$y_k = T_k \left(y_{k-1} + x_{k-1} - z_{k-1} - \frac{\beta}{1 - \beta} x^* \right) \quad (10)$$

where T_k is the mapping as follows.

$$T_k(x)(t) = (1 - \beta)x(t) - (1 - \beta)e^{-\Gamma_k t} \int_0^t e^{\Gamma_k s} \Gamma_k x(s) ds \quad (11)$$

For the sake of simple notation, we will write (11) for short as $T_k(x) = (1 - \beta)x - (1 - \beta)e^{-\Gamma_k t} \int_0^t e^{\Gamma_k s} \Gamma_k x ds$ by abusing T_k a little. Note that the mapping above is k -dependent because of Γ_k is k -dependent. Now we will give the result of contraction mapping for k -dependent case, which differs from pp. 655 [25] (where only k -invariant mappings are studied).

An operator T between two real linear spaces \mathcal{X} and \mathcal{Y} is called a *linear mapping* or *linear operator* if $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathcal{X}$. There is a norm $\|\bullet\|$ defined on \mathcal{X} and \mathcal{Y} . Then, a linear mapping is *bounded* if there exists a constant $M \geq 0$ such that

$$\|T(x)\| \leq M\|x\| \text{ for all } x \in \mathcal{X}.$$

The sets consisted of all these bounded linear mapping T is denoted by $B(\mathcal{X}, \mathcal{Y})$. We write $B(\mathcal{X})$ for short if the domain and range spaces are the same. Moreover, the mapping norm is defined as

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}.$$

Definition 4.1 (Uniformly convergence (pp.109, [26])): If $\{T_k\}$ is a sequence of mappings in $B(\mathcal{X}, \mathcal{Y})$ and

$$\lim_{k \rightarrow \infty} \|T_k - T\| = 0$$

for some $T \in B(\mathcal{X}, \mathcal{Y})$, then we say that T_k converges uniformly to T .

Note that $\|T_k(x) - T(x)\| \leq \|T_k - T\|\|x\|$. Given that $\|x\|$ is bounded, $\lim_{k \rightarrow \infty} \|T_k - T\| = 0$ implies that $\lim_{k \rightarrow \infty} T_k(x) = T(x)$. In other words, uniform convergence implies *strong convergence*.

Theorem 4.1 (k -dependent contraction mapping): Let S be a compact subset of a Banach space \mathcal{X} . If a mapping sequence $\{T_k\}$ satisfies

- C1) for every k , $T_k \in B(S)$;
- C2) $\|T_k(x) - T_k(y)\| \leq \rho\|x - y\|, \forall x, y \in S$, for a universal *contraction mapping ratio* $\rho \in [0, 1)$, denoting \mathcal{T} as the set consisted of all such T_k ;
- C3) T_k converges uniformly to T_∞ as $k \rightarrow \infty$, for some $T_\infty \in \mathcal{T}$,

then T_k is called a *k -dependent contraction mapping* on S . Furthermore,

- there exists a unique solution $x^* \in S$ satisfying $x^* = T_\infty(x^*)$;
- x^* is independent of the initial value $x_1 \in S$.

Proof: The proof is similar to Theorem B.1 in [25]. Arbitrarily select $x_1 \in S$ and generate a sequence $\{x_k\}$ according to the formula $x_{k+1} = T_k(x_k)$. Every $x_k \in S$, since $T_k \in B(S)$. First, we will show $\{x_k\}$ is a Cauchy sequence.

It follows from S is compact that there is a constant $D > 0$ such that $\|x\| \leq D$ for all $x \in S$. Additionally, that $\{T_k\}$ is a Cauchy sequence follows, since T_k converges uniformly. Then, for an arbitrary $\epsilon > 0$, there exists a k_ϵ such that $\|T_m - T_n\| \leq \epsilon/D$ for any $m, n \geq k_\epsilon$. For $k - 1 > k_\epsilon$, we have

$$\begin{aligned} \|x_{k+1} - x_k\| &= \|T_k(x_k) - T_{k-1}(x_{k-1})\| \\ &\leq \|T_k(x_k) - T_k(x_{k-1})\| \\ &\quad + \|T_k(x_{k-1}) - T_{k-1}(x_{k-1})\| \\ &\leq \rho \|x_k - x_{k-1}\| + \|T_k - T_{k-1}\| \|x_{k-1}\| \\ &\leq \rho \|x_k - x_{k-1}\| + \epsilon \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{k+r} - x_k\| &\leq \sum_{i=0}^{r-1} \|x_{k+i+1} - x_{k+i}\| \\ &\leq \sum_{i=0}^{r-1} [\rho^{i+1} \|x_k - x_{k-1}\| + \rho^i \epsilon] \\ &\leq \frac{\rho}{1-\rho} \|x_k - x_{k-1}\| + \frac{\epsilon}{1-\rho} \\ &\leq \frac{\rho}{1-\rho} (\rho \|x_{k-1} - x_{k-2}\| + \epsilon) + \frac{\epsilon}{1-\rho} \\ &\leq \frac{\rho}{1-\rho} \left(\rho^{k-k_\epsilon-1} \|x_{k_\epsilon+1} - x_{k_\epsilon}\| + \sum_{i=0}^{k-k_\epsilon-2} \rho^i \epsilon \right) \\ &\quad + \frac{\epsilon}{1-\rho} \\ &\leq \frac{2\rho^{k-k_\epsilon} D}{1-\rho} + \frac{\rho\epsilon}{(1-\rho)^2} + \frac{\epsilon}{1-\rho} \\ &\leq \frac{2\rho^{k-k_\epsilon} D}{1-\rho} + \frac{\epsilon}{(1-\rho)^2} \end{aligned}$$

Since ϵ is chosen arbitrarily, the right hand side will go to 0 as $k \rightarrow \infty$. Hence, $\{x_k\}$ is a Cauchy sequence. From that \mathcal{X} is complete, $x_k \rightarrow x^* \in \mathcal{X}$ as $k \rightarrow \infty$. Furthermore, S is closed; it follows that $x^* \in S$.

The second step is to prove $x^* = T_\infty(x^*)$. For any $x_{k+1} = T_k(x_k)$, it can be obtained that

$$\begin{aligned} \|x^* - T_\infty(x^*)\| &\leq \|x^* - x_{k+1}\| + \|x_{k+1} - T_\infty(x^*)\| \\ &\leq \|x^* - x_{k+1}\| + \|x_{k+1} - T_\infty(x_k)\| \\ &\quad + \|T_\infty(x_k) - T_\infty(x^*)\| \\ &\leq \|x^* - x_{k+1}\| + \|T_k - T_\infty\| \|x_k\| \\ &\quad + \rho \|x_k - x^*\| \end{aligned}$$

It is apparent that the right hand side can be made arbitrarily small by selecting a sufficiently large k . Thereafter, $x^* = T_\infty(x^*)$.

Finally, we will show the uniqueness. Suppose that there is another fix point y^* satisfying $y^* = T_\infty(y^*)$. Then, we have

$$\|x^* - y^*\| \leq \|T_\infty(x^*) - T_\infty(y^*)\| \leq \rho \|x^* - y^*\|$$

Since ρ is strictly less than 1, it is a contradiction; then $x^* = y^*$. This completes the whole proof. ■

Define a Banach space $\mathcal{X} = C[0, L]$ and a compact set

$$S = \{y \in \mathcal{X} \mid \|y\|_\lambda \leq D_2\} \quad (12)$$

It implies that z_k is contained in $\mathcal{S}_z = \{z \in \mathcal{X} \mid \|z - x^*\|_\lambda \leq D_2\}$. The lemma below shows the conditions to fulfill C2 for T_k .

Lemma 4.1: Let A1-A3 hold. For arbitrary $\beta \in (0, 1)$, there is a λ_0 such that $\|T_k(x) - T_k(y)\|_\lambda \leq \rho \|x - y\|_\lambda$ for any $x, y \in \mathcal{X}$, $\rho \in (0, 1)$ for every $\lambda \in (\lambda_0, \infty)$. Moreover, ρ and λ_0 are independent of k .

Proof: For details of the proof, please refer to Appendix A. ■

Remark 4.3: Many ILC literatures do not use such a forgetting factor, since its existence will comprise the zero tracking error (perfect tracking) iteration-wisely [18]. However, the above lemma suggests otherwise in our case. It is necessary to introduce such a forgetting factor β to ensure the existence of the k -dependent contraction mapping, since that the perfect tracking cannot be achieved due to the existence of dither signal. If without β , this undesired effect would keep accumulating and the overall performance would deteriorate rapidly.

Remark 4.4: If the C-norm is used instead of λ -norm, or equivalently $\lambda = 0$, then according to the proof of Lemma 4.1, β has to be greater than $2 - \sqrt{2}$. It suggests that λ -norm somehow enlarges the feasible basin of β .

Remark 4.5: From (11), it is evident that the mapping T_k is a linear mapping. Thus, we can rewrite (10) as

$$y_k = T_k(y_{k-1}) + T_k(x_{k-1} - z_{k-1}) + T_k\left(-\frac{\beta}{1-\beta}x^*\right) \quad (13)$$

In (13), it shows that the second and third terms on the right hand side play a role like “disturbances”; the second one is caused by the approximation, while the third is caused by the forgetting factor. The two “disturbances” are still different: $T_k(x_{k-1} - z_{k-1})$ is a persistently active noise, while $T_k[-\beta x^*/(1-\beta)]$ is just a constant offset.

A. I-type ILC

To this end, we are ready to present our contribution on ILC, which studies the behavior of the disturbance-free MLB system (8), i.e.,

$$z_k = (1 - \beta)z_{k-1} - \Gamma_k \int_0^t (z_k - x^*) ds \quad (14)$$

Or equivalently, its corresponding error system

$$y_k = T_k(y_{k-1}) + T_k\left(-\frac{\beta}{1-\beta}x^*\right) \quad (15)$$

will be studied. It should be mentioned that an underlying assumption has been imposed, namely, x^* is known, when we discuss on I-type ILC. Now, we define another mapping as

$$G_k(x) \triangleq T_k(x) + T_k\left(-\frac{\beta}{1-\beta}x^*\right)$$

The objective is to show G_k is a k -dependent contraction mapping. It is easy to verify that G_k satisfies C2, since that $G_k(x) - G_k(y) = T_k(x) - T_k(y)$ and $T_k(x) - T_k(y)$ fulfills C2. The following lemma gives a sufficient condition that G_k fulfills C1.

Lemma 4.2: Let A1-A3 be satisfied. Given $\lambda > 0, \beta \in (0, 1), \rho \in (0, 1)$, if D_2 in (12) satisfies

$$D_2 \geq \max\{D_0, D^*\},$$

then G_k maps S into S . D^* is defined as

$$D^* = \frac{\beta\rho}{(1-\beta)(1-\rho)} \|x^*\|_\lambda$$

And $D_0 = \|y_1\|_\lambda$, y_1 yielded by

$$y_1 = -x^* - \Gamma_1 \int_0^t y_1 ds$$

It is simply executing (15) on $k = 1$.

Proof: For details of the proof, please refer to Appendix B. ■

Lemma 4.2 not only presents a sufficient condition for G_k satisfying C1, but also states that $\|y_k\|_\lambda$ is uniformly bounded. Obviously, we can offer more than that, i.e., convergence and uniqueness of the limit solution, by invoking the k -dependent contraction mapping theorem.

Theorem 4.2: A1-A3 are satisfied. If $D_2 \geq \max\{D_0, D^*\}$ as in Lemma 4.2, there exists a unique limit solution for y_k in (15) as k tends to infinity. Moreover, $\lim_{k \rightarrow \infty} y_k = y_\infty$, where y_∞ is defined as the solution to the following equation.

$$y_\infty = T_\infty \left(y_\infty - \frac{\beta}{1-\beta} x^* \right) \quad (16)$$

T_∞ is the limit of T_k as k tends to infinity, like

$$T_\infty(x) = (1-\beta)x - (1-\beta)e^{-\Gamma_\infty t} \int_0^t e^{\Gamma_\infty s} \Gamma_\infty x ds \quad (17)$$

with

$$\Gamma_\infty = \frac{\alpha Q}{2\beta} \quad (18)$$

Proof: It is noted from (16) that (16) is equivalent to

$$y_\infty = G_\infty(y_\infty)$$

Hence, we only need to check whether G_k satisfies C1-C3 or not. Since \mathcal{X} is equipped with λ -norm, we can define the mapping norm by the λ -norm as

$$\|G_k\| = \sup_{\|x\|_\lambda=1} \|G_k(x)\|_\lambda$$

Expanding $G_k(x)$, we have

$$\|G_k\| \leq \sup_{\|x\|_\lambda=1} \|T_k(x)\|_\lambda + \left\| T_k \left(-\frac{\beta}{1-\beta} x^* \right) \right\|_\lambda$$

On the other hand, from Lemma 4.1, it is known that $\|T_k(x)\|_\lambda = \|T_k(x) - T_k(0)\|_\lambda \leq \rho \|x\|_\lambda$. It immediately follows that

$$\|G_k\| \leq \rho + \left\| T_k \left(-\frac{\beta}{1-\beta} x^* \right) \right\|_\lambda < +\infty$$

Combining with Lemma 4.2, $G_k \in B(S)$ for arbitrary k . Also noted from Lemma 4.1, the mapping sequence $\{G_k\}$ satisfies C1 and C2; it remains to show that G_k converges to G_∞

uniformly. It suffices to show T_k converges to T_∞ uniformly. By definition, we have

$$\|T_k - T_\infty\| = \sup_{\|x\|_\lambda=1} \|T_k(x) - T_\infty(x)\|_\lambda$$

From the definition of λ -norm and (10), we further get that

$$\begin{aligned} \|T_k - T_\infty\| &\leq (1-\beta) \\ &\times \sup_{\substack{t \in [0, L] \\ \|x\|_\lambda=1}} e^{-\lambda t} \left\| \int_0^t \left(e^{-\Gamma_k(t-s)} \Gamma_k - e^{-\Gamma_\infty(t-s)} \Gamma_\infty \right) x ds \right\|_\infty \end{aligned}$$

By the mean value theorem, it is easy to derive that

$$\begin{aligned} \|T_k - T_\infty\| &\leq (1-\beta) \sup_{\substack{t \in [0, L] \\ \|x\|_\lambda=1}} e^{-\lambda t} \left\| \int_0^t \left(e^{-\Gamma_k(t-s)} \Gamma_k - e^{-\Gamma_\infty(t-s)} \Gamma_\infty \right) ds \right\|_\infty \\ &\quad \times \|x(\xi)\|_\infty, (\xi \in [0, L]) \\ &\leq (1-\beta) \sup_{t \in [0, L]} \left\| \int_0^t \left(e^{-\Gamma_k(t-s)} \Gamma_k - e^{-\Gamma_\infty(t-s)} \Gamma_\infty \right) ds \right\|_\infty \\ &\leq (1-\beta) \sup_{t \in [0, L]} \left\| e^{-\Gamma_k t} - e^{-\Gamma_\infty t} \right\|_\infty \end{aligned}$$

The term $\exp(-\Gamma_k t)$ can approach $\exp(-\Gamma_\infty t)$ arbitrarily small as $k \rightarrow \infty$. Therefore, it can be concluded that

$$\|T_k - T_\infty\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

which is the uniform convergence. Then, apply the k -dependent contraction mapping, we can conclude the result. ■

Remark 4.6: Theorem 4.2 shows that the trajectory of the disturbance-free MLB system (14) will ultimately converge to a fixed trajectory, which is parameterized by β and x^* . From (17), it is clear that 0 is a solution to the equation $x = T_\infty(x)$. From (16), one can see that y_∞ will approach 0 if $\beta \rightarrow 0$. But will y_∞ be 0 if $\beta = 0$? From Lemma 4.1, it is known that the contraction mapping method will fail when $\beta = 0$. Thus, we use another way to prove that conjecture in the following theorem.

Before presenting the theorem, we have to make a remark on this particular case $\beta = 0$. In the following theorem, we will abandon Γ_k but use a constant gain Γ (positive definite matrix, not necessary restricting to $\alpha Q/2$) instead. The reasons for doing so are twofold. First, (6) suggests that Γ_k will be ill-defined, since $\gamma_k \rightarrow \infty$. Second, as Remark 4.3 states, the motivation of introducing β is to handle the “control input disturbance”, and the gain Γ_k becomes k -varying because of β ; now we are hereby dealing with “disturbance”-free control system.

Theorem 4.3: Considering $\beta = 0$, that is z_k is generated by the following formula

$$\dot{z}_k = \dot{z}_{k-1} - \Gamma(z_k - x^*). \quad (19)$$

If A1-A3 hold and $x^*(0) = 0$, then

$$z_k \rightarrow x^* \text{ almost everywhere as } k \rightarrow \infty.$$

Proof: From (19), within this proof, we are equivalently studying

$$\dot{y}_k = \dot{y}_{k-1} - \Gamma y_k \quad (20)$$

Define an index as follows.

$$J_k = \int_0^L e^{-\lambda t} \dot{y}_k^T \dot{y}_k dt \quad (21)$$

where $\lambda > 0$. Rewriting (20) as $\dot{y}_{k-1} = \dot{y}_k + \Gamma y_k$, we insert it into (21) and compare the difference of (21) between k and $k-1$.

$$\begin{aligned} J_k - J_{k-1} &= \int_0^L e^{-\lambda t} [\dot{y}_k^T \dot{y}_k - (\dot{y}_k + \Gamma y_k)^T (\dot{y}_k + \Gamma y_k)] dt \\ &= - \int_0^L e^{-\lambda t} y_k^T \Gamma^T \Gamma y_k dt - 2 \int_0^L e^{-\lambda t} y_k^T \Gamma \dot{y}_k dt \end{aligned}$$

As for the second term in the right hand side, we integrate by parts and derive that

$$2 \int_0^L e^{-\lambda t} y_k^T \Gamma \dot{y}_k dt = e^{-\lambda t} y_k^T \Gamma y_k \Big|_{t=0}^{t=L} + \lambda \int_0^L e^{-\lambda t} y_k^T \Gamma y_k dt$$

Thus, we have

$$\begin{aligned} J_k - J_{k-1} &= - \int_0^L e^{-\lambda t} y_k^T (\Gamma^T \Gamma + \lambda \Gamma) y_k dt \\ &\quad - e^{-\lambda t} y_k^T (L) \Gamma y_k (L) \\ &\leq - \rho_{\min} \int_0^L e^{-\lambda t} y_k^T y_k dt \end{aligned}$$

Let ρ_{\min} be the smallest eigenvalue of the matrix $\Gamma^T \Gamma + \lambda \Gamma$. It is apparent that $\rho_{\min} > 0$ since Γ is positive definite and $\lambda > 0$. Thus, $\{J_k\}$ is a non-increasing real number sequence, and J_k is bounded below by 0, J_k converges. It follows that

$$\lim_{k \rightarrow \infty} \left(\int_0^L y_k^T y_k dt \right)^{\frac{1}{2}} = 0$$

It is the L_2 -norm of y_k . Thus, we can claim that y_k converges to 0 almost everywhere. ■

Remark 4.7: Theorem 4.3 gives a weaker result than Theorem 4.2, since it can converge to 0 except on a set whose measure is 0 and the uniqueness is lost. The result is quite understandable from a perspective of Laplace transform. Taking Laplace transform on both sides of (20), we have

$$Y_k(s) = \frac{s}{s + \Gamma} Y_{k-1}(s)$$

The modulus of the gain is less than 1, i.e., $|s/(s + \Gamma)| < 1$. But it tends to 1 as $s \rightarrow \infty$. It suggests that this algorithm have a weaker decaying effect on high-frequency signal. It coincides with the result.

B. Iterative learning extremum seeking (ILES)

We follow the similar idea to study the dynamics of ILES. ILES is an ILC control policy with “control input disturbance”. Due to the “disturbance”, the unique limit solution cannot be achieved. Therefore, ILES converging to a set will be shown instead, if the “disturbance” is bounded.

According to (13), we define a new operator H_k as

$$H_k(x) \triangleq T_k(x) + T_k[-\beta x^*/(1 - \beta)] + T_k(x_{k-1} - z_{k-1})$$

Since T_k is a bounded operator, so is H_k provided that $x_{k-1} - z_{k-1}$ is bounded. It is obvious that $H_k(x) - H_k(y) = T_k(x) -$

$T_k(y)$; it is easy to verify that H_k satisfies C2. Following the procedures we did in Section IV.A, we are going to show that how to properly design D_2 to ensure H_k maps S into S .

The following lemma lays the foundation of inductive arguments towards that conclusion. It basically means that for any k we can always ensure that x_k is in a invariant set given that its MLB system z_k is in the interior of the invariant set, if x_1, x_2, \dots, x_{k-1} are all in that set. This can be achieved by selecting a frequency ω larger than a threshold ω_0 , which is independent of k and t .

Lemma 4.3: Let A1-A3 be satisfied and suppose that $x_1, x_2, \dots, x_{k_0-1}$ are uniformly in t contained within a compact set $\mathcal{S} \in \mathbb{R}^n$ and $0 \in \text{int } \mathcal{S}$. If z_{k_0} is contained in $\mathcal{K} \subset \text{int } \mathcal{S}$, $0 \in \mathcal{K}$, then there exists a $\omega_0 \in (0, +\infty)$ such that for every $\omega_0 \in (\omega_0, \infty)$, $x_{k_0}(t)$ is uniformly in t contained within \mathcal{S} as well for any $\beta \in (0, 1)$. Moreover, $\text{dist}(\mathcal{K}, \mathcal{S})$ can be made arbitrarily small by selecting a sufficiently large ω .

Proof: This proof uses the similar arguments as Theorem 1 in [24], but tailors them for the iterative case. For details of the proof, please refer to the Appendix C. ■

The lemma above also indicates that the “disturbance” $\|x_k - z_k\|_\lambda$ is uniformly bounded; it can be arbitrarily small by selecting a sufficiently large ω .

Theorem 4.4: Let A1-A3 hold. Given λ and $\rho \in (0, 1)$, there exists a $\omega_0 \in (0, +\infty)$ such that for every $\omega \in (\omega_0, \infty)$, H_k maps S into S , if D_2 in (12)

$$D_2 \geq \max\{D_0, D^*\}$$

where D_0 is defined in Lemma 4.2 and

$$D^* = \frac{\rho}{1 - \rho} \left(D_1 + \frac{\beta}{1 - \beta} \|x^*\|_\lambda \right).$$

D_1 is the uniform bound of $\|x_k - z_k\|_\lambda$ for an arbitrary k .

Proof: We are going to show the theorem in an inductive way.

First, at $k = 1$, both the extremum seeking system (4) and the MLB system (5) are in the standard form. From (9), the explicit solution of the MLB system (5) is

$$z_1 = \int_0^t e^{-\Gamma_1(t-s)} \Gamma_1 x^* ds$$

Because all the eigenvalues of $-\Gamma_1$ lie in the open left complex plane and x^* is contained in a compact set, z_1 is well defined for initial condition $z_1(0) = 0$ in a compact set over the time interval $[0, L]$. According to the definition of D_2 , it is known that $\|y_1\|_\lambda < D_2$. Thus, by Lie bracket theorem [24], it is known that the distance between x_1 and z_1 can be arbitrarily small provided that the frequency ω is sufficiently large. Hence, there exists a ω_1 such that for every $\omega \in (\omega_1, \infty)$, we have $\|x_1 - z_1\|_\lambda < D_1$.

Now, we assume that $\|y_i\|_\lambda < D_2$ and $\|x_i - z_i\|_\lambda < D_1$ for any $i = 1, 2, \dots, k-1$ over the entire time interval. Then, we will show that these two relations are still valid for $i = k$. From Lemma 4.3, it is easy to conclude that there exists a ω_2 such that for every $\omega \in (\omega_2, +\infty)$, $\|x_k - z_k\|_\lambda < D_1$. We only need to show $\|y_k\|_\lambda < D_2$.

By the linearity of T_k , we have

$$\begin{aligned}
\|y_k\|_\lambda &= \left\| H_k(y_{k-1}) - H_k(0) \right. \\
&\quad \left. + T_k \left(x_{k-1} - z_{k-1} - \frac{\beta}{1-\beta} x^* \right) \right\|_\lambda \\
&\leq \left\| T_k \left(x_{k-1} - z_{k-1} - \frac{\beta}{1-\beta} x^* \right) - T_k(0) \right\|_\lambda \\
&\quad + \|H_k(y_{k-1}) - H_k(0)\|_\lambda \\
&\leq \rho \|y_{k-1}\|_\lambda + \rho \left\| x_{k-1} - z_{k-1} - \frac{\beta}{1-\beta} x^* \right\|_\lambda \\
&\leq \rho D_2 + \rho \left(D_1 + \frac{\beta}{1-\beta} \|x^*\|_\lambda \right) \leq D_2
\end{aligned}$$

Therefore, by selecting $\omega_0 = \max\{\omega_1, \omega_2\}$, we have $\|y_k\|_\lambda < D_2$ for all k . This completes the proof. ■

Theorem 4.4 suggests that the uniform bound of tracking error of the MLB system (5) – D_2 , can be made small by reducing the approximation error (D_1), i.e., employing a sufficiently large ω .

Since $x_k - z_k$ is consistently varying, it is impossible to show that H_k satisfies C3. Therefore, we cannot conclude the unique limit solution; however, we can show that y_k converges to a λ -norm ball.

Theorem 4.5: A1-A3 are satisfied. Given $\lambda, \beta \in (0, 1)$ and $\rho \in (0, 1)$, there exists a ω_0 such that for every $\omega \in (\omega_0, \infty)$, y_k in (13) will converge to a set \mathcal{Y} as k tends to infinity. Furthermore,

$$\mathcal{Y} = \{y \in S \mid \|y - y_\infty\|_\lambda \leq D_y\} \quad (22)$$

Here y_∞ is defined in (16), $D_y = \rho D_1 / (1 - \rho)$; D_1 is the uniform bound of $\|x_k - z_k\|_\lambda$ for an arbitrary k .

Proof: The proof is in Appendix D. ■

Remark 4.8: Theorem 4.5 suggests that we cannot achieve “perfect tracking” or a fixed limit trajectory unlike many ILC control laws, due to the existence of dither signals (sinusoid signals). However, according to Lemma 4.3, D_1 can be made arbitrarily small for a sufficiently large ω . Thus, so is D_y , since D_y is proportional to D_1 ; that means we can make the ultimate trajectory be as close to a fixed limit trajectory as one wishes by selecting a sufficiently large frequency. In the mean time, we can also let the fixed trajectory be as close to zero as possible by having a small enough forgetting factor β .

V. ILLUSTRATIVE EXAMPLE

A. I-type ILC

Consider the following tracking reference for the ILC control law in (14).

$$x^* = -\sin\left(\frac{\pi}{20}t\right)$$

Fig. 2 shows the trajectory evolution versus iteration for the I-type ILC with a forgetting factor $\beta = 0.5$. It clearly verifies that the trajectory (z_k) will converge to a fixed trajectory, but the trajectory has a gap with the tracking reference. Fig. 3 demonstrates the evolution for the I-type ILC without the forgetting factor β . It is shown that the trajectory will converge to the reference ultimately, thus “perfect tracking” achieved. Meanwhile, it proves the result of Theorem 4.3.

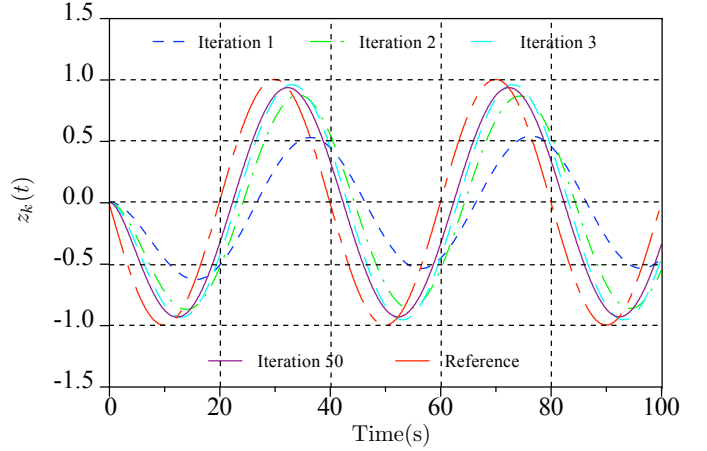


Fig. 2: Integral-type ILC with forgetting factor $\beta = 0.5$

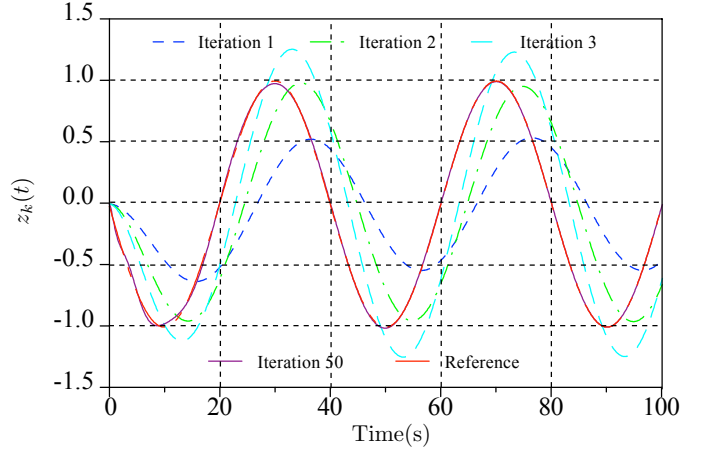


Fig. 3: Integral-type ILC without forgetting factor $\beta = 0$

B. ILES

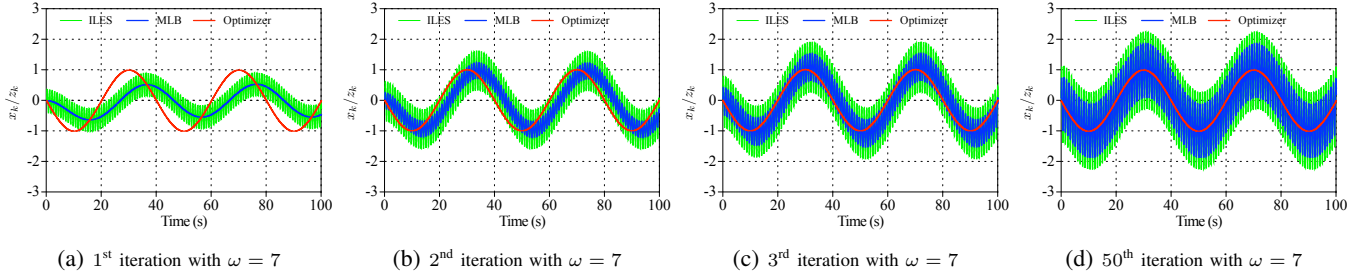
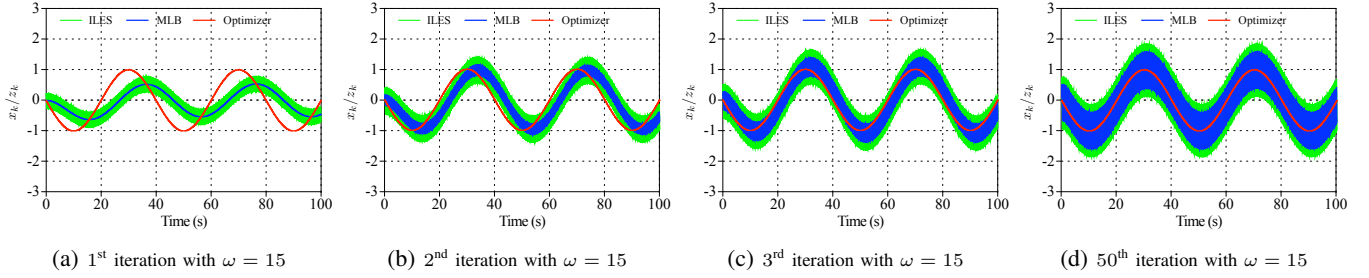
We study the following static map.

$$y = x^2 + 2 \sin\left(\frac{\pi}{20}t\right) x$$

It is evident that the minimizer trajectory is $x^* = -\sin\left(\frac{\pi}{20}t\right)$. As for the iterative learning extremum seeking control law, α is selected as 0.1 and the forgetting factor $\beta = 0.3$. Figs. 4a-4d shows the evolutions of the original system and the MLB system (5) in the 1st, 2nd, 3rd, 50th iterations under the frequency $\omega = 7$. They indicate that the tracking performances of both the systems improve gradually, although the fluctuations are getting larger but finally bounded. Figs. 5a-5d shows the similar evolutions under the frequency $\omega = 15$. Comparing Figs. 4a-4d and Figs. 5a-5d, it implies that both the tracking error (the fluctuation of the MLB system (5)) and the approximation error (the gap between the original system and the MLB system (5)) are smaller under a larger frequency ω .

VI. CONCLUSION & OUTLOOK

This paper has proposed an iterative learning extremum seeking approach to solve the time varying optimizer tracking

Fig. 4: The evolution of the original system and MLB system under frequency $\omega = 7$.Fig. 5: The evolution of the original system and MLB system under frequency $\omega = 15$.

problem. A modified Lie bracket system has been introduced to study the behavior of the ILES system. It has shown that the MLB system is an online integral-type ILC control law with bounded “control input disturbance”. The convergence of the corresponding ILC control law has been analyzed. Based on that, the convergence of the proposed ILES to a set has been shown. The size of the set is reducible by tuning the frequency of the dither signal. The distance between set’s center and original is also tunable by some appropriate forgetting factors β . In the future, it is quite interesting to investigate how to extend the method to cover a general function $F(x, t)$. It is worthy to study the dynamic mapping situation as well.

APPENDIX A PROOF OF LEMMA 4.1

Proof: From the definitions of T_k and λ -norm, we have

$$\begin{aligned} & \|T_k(x) - T_k(y)\|_\lambda \\ &= \max_{t \in [0, L]} e^{-\lambda t} (1 - \beta) \\ & \quad \left\| (x - y) - e^{-\Gamma_k t} \int_0^t e^{\Gamma_k s} \Gamma_k (x - y) ds \right\|_\infty \end{aligned}$$

It follows from the triangle inequality of norm and $\max\{a + b\} \leq \max\{a\} + \max\{b\}$ that

$$\begin{aligned} & \|T_k(x) - T_k(y)\|_\lambda \\ & \leq \max_{t \in [0, L]} e^{-\lambda t} (1 - \beta) \|x - y\|_\infty \\ & \quad + \max_{t \in [0, L]} e^{-\lambda t} (1 - \beta) \\ & \quad \left\| e^{-\Gamma_k t} \int_0^t e^{\Gamma_k s} \Gamma_k (x - y) ds \right\|_\infty \end{aligned}$$

The first term on the right hand side is exactly $(1 - \beta)\|x - y\|_\lambda$ according to the definition of λ -norm. For the second term, we do the following operation.

$$\begin{aligned} & \|T_k(x) - T_k(y)\|_\lambda \leq (1 - \beta)\|x - y\|_\lambda + \max_{t \in [0, L]} e^{-\lambda t} (1 - \beta) \\ & \quad \left\| e^{-\Gamma_k t} \int_0^t e^{(\Gamma_k + \lambda I)s} e^{-\lambda s} \Gamma_k (x - y) ds \right\|_\infty \end{aligned}$$

According to mean value theorem, it can be obtained that

$$\begin{aligned} & \left\| e^{-\Gamma_k t} \int_0^t e^{(\Gamma_k + \lambda I)s} e^{-\lambda s} \Gamma_k (x - y) ds \right\|_\infty \\ &= \left\| e^{-\Gamma_k t} \int_0^t e^{(\Gamma_k + \lambda I)s} \Gamma_k ds \left(e^{-\lambda \xi} (x(\xi) - y(\xi)) \right) \right\|_\infty, \\ & \quad [\xi \in (0, t)] \\ & \leq e^{-\lambda \xi} \|x(\xi) - y(\xi)\|_\infty \left\| e^{-\Gamma_k t} \int_0^t e^{(\Gamma_k + \lambda I)s} \Gamma_k ds \right\|_\infty, \\ & \quad [\xi \in (0, t)] \\ & \leq \max_{s \in [0, t]} \{e^{-\lambda s} \|x - y\|_\infty\} \left\| e^{-\Gamma_k t} \int_0^t e^{(\Gamma_k + \lambda I)s} \Gamma_k ds \right\|_\infty \end{aligned}$$

The first inequality is followed from the definition of induced matrix norm. It is also noted that

$$\max_{s \in [0, t]} \{e^{-\lambda s} \|x - y\|_\infty\} \leq \|x - y\|_\lambda$$

Hence,

$$\begin{aligned}
& \|T_k(x) - T_k(y)\|_\lambda \\
& \leq (1 - \beta) \|x - y\|_\lambda \left(1 + \max_{t \in [0, L]} e^{-\lambda t} \right. \\
& \quad \left. \left\| e^{-\Gamma_k t} \int_0^t e^{(\Gamma_k + \lambda I)s} \Gamma_k ds \right\|_\infty \right) \\
& \leq (1 - \beta) \|x - y\|_\lambda \left(1 + \max_{t \in [0, L]} \left\| \int_0^t e^{(\Gamma_k + \lambda I)(s-t)} \Gamma_k ds \right\|_\infty \right) \\
& \leq (1 - \beta) \|x - y\|_\lambda \left[1 + \max_{t \in [0, L]} \left\| \left(I - e^{-(\Gamma_k + \lambda I)t} \right) \Gamma_k \right. \right. \\
& \quad \left. \left. (\Gamma_k + \lambda I)^{-1} \right\|_\infty \right]
\end{aligned}$$

For a matrix $A \in \mathbb{M}_n$, $\frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_\infty \leq \sqrt{n} \|A\|_2$, which is obtained from the norm equivalence theorem. Then, it follows that

$$\begin{aligned}
& \left\| \left(I - e^{-(\Gamma_k + \lambda I)t} \right) \Gamma_k (\Gamma_k + \lambda I)^{-1} \right\|_\infty \\
& \leq \sqrt{n} \left\| \left(I - e^{-(\Gamma_k + \lambda I)t} \right) \Gamma_k (\Gamma_k + \lambda I)^{-1} \right\|_2 \\
& \leq \sqrt{n} \left\| \Gamma_k (\Gamma_k + \lambda I)^{-1} \right\|_2
\end{aligned}$$

Thereafter, we have

$$\|T_k(x) - T_k(y)\|_\lambda \leq (1 - \beta) \|x - y\|_\lambda (1 + \sqrt{n} \|\Gamma_k (\Gamma_k + \lambda I)^{-1}\|_2)$$

Note that $\|\Gamma_k (\Gamma_k + \lambda I)^{-1}\|_2 = \|I - (\Gamma_k + \lambda I)^{-1}\|_2$, it is a non increasing function of Γ_k . From the assumption, it is known that $\Gamma_k \geq \alpha\delta/2I$. Denoting $\rho = (1 - \beta)(1 + \sqrt{n}\alpha\delta/(\alpha\delta + 2\lambda))$, it is easy to see there always exists a $\rho < 1$ when $\lambda \in (\lambda_0, +\infty)$ for arbitrary $\beta \in (0, 1)$ with

$$\lambda_0 = \max \left\{ 0, \frac{\alpha\delta[\sqrt{n}(1 - \beta) - \beta]}{2\beta} \right\}$$

This completes the proof. \blacksquare

APPENDIX B PROOF OF LEMMA 4.2

Proof: We are going to show the statement by inductive arguments.

When $k = 1$, from (9), the explicit solution of z_1 is

$$z_1 = - \int_0^t e^{-\Gamma_1(t-s)} \Gamma_1 x^* ds$$

Since all the eigenvalues associated with Γ_1 locates in the right complex plane and x^* is continuous over the interval $[0, L]$, z_1 is well defined and bounded over $[0, L]$. Hence, $\|y_1\|_\lambda$ is bounded. Furthermore, it is evident that $\|y_1\|_\lambda \leq D_2$, equivalently, $y_1 \in S$.

Assume that $y_{k-1} \in S$ or $\|y_{k-1}\|_\lambda \leq D_2$ holds for the $k - 1$ -th iteration. It remains to show that $y_k \in S$.

From (15) and Lemma 4.1, we have

$$\begin{aligned}
\|y_k\|_\lambda &= \|G_k(y_{k-1})\|_\lambda \\
&\leq \left\| T_k(y_{k-1}) - T_k(0) + T_k \left(-\frac{\beta}{1 - \beta} x^* \right) - T_k(0) \right\|_\lambda \\
&\leq \|T_k(y_{k-1}) - T_k(0)\|_\lambda \\
&\quad + \left\| T_k \left(-\frac{\beta}{1 - \beta} x^* \right) - T_k(0) \right\|_\lambda \\
&\leq \rho \|y_{k-1}\|_\lambda + \rho \left\| \frac{\beta}{1 - \beta} x^* \right\|_\lambda \\
&\leq \rho D_2 + \rho \left\| \frac{\beta}{1 - \beta} x^* \right\|_\lambda
\end{aligned}$$

From the statement, we have $D_2 \geq \frac{\beta\rho}{(1-\beta)(1-\rho)} \|x^*\|_\lambda$. Thus,

$$\|y_k\|_\lambda \leq \rho D_2 + \rho \left\| \frac{\beta}{1 - \beta} x^* \right\|_\lambda \leq \rho D_2 + (1 - \rho) D_2 = D_2$$

Hence, $G_k(y_{k-1}) \in S$, and this completes the proof. \blacksquare

APPENDIX C PROOF OF LEMMA 4.3

Proof: The proof is adapted from Theorem 1 in [24]. The major differences are: first, [24] dealing with more general form – input affine system, we only focus on sinusoid input signal; second, [24] suitable for infinite time horizon and single iteration, we are handling the case of finite time horizon and multiple iterations.

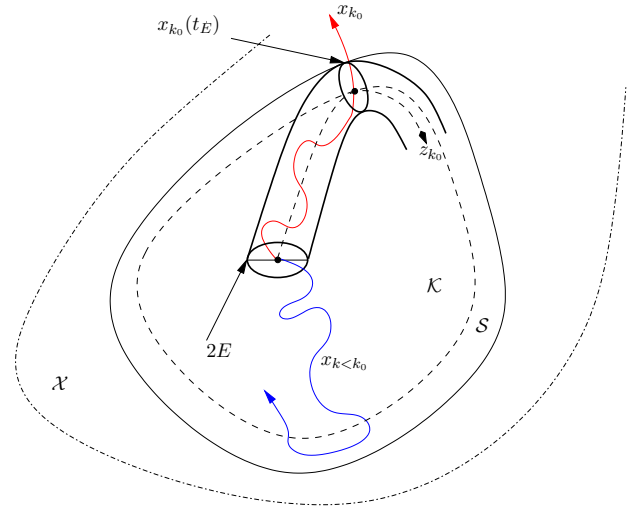


Fig. 6: The illustration of the proof idea

The basic idea to show the lemma is illustrated in Fig. 6. $x_{k < k_0}$ are within S and z_{k_0} is within K . We construct a tube with radius E along the trajectory of z_{k_0} . If $E \leq \text{dist}(K, S)$ and x_{k_0} is within the tube, it can conclude that $x_{k_0} \in S$. Moreover, supposing the x_{k_0} leaves the tube at arbitrary time t_E , if we can show t_E is not the time that x_{k_0} leaves the tube, then we can claim that x_{k_0} will never leave the tube.

Evaluating (4), (5) at k_0 , Subtracting (5) from (4) and integrating on both sides, we have

$$\begin{aligned} x_{k_0} - z_{k_0} = & -\alpha \int_0^t F(x_{k_0}, s) \sqrt{\omega} \sin(\omega s) ds \\ & + \int_0^t \sqrt{\omega} \cos(\omega s) ds \\ & + \frac{\alpha \gamma_{k_0}}{2} \int_0^t \nabla_z F(z_{k_0}, s) ds \end{aligned} \quad (23)$$

Let R_a be the first term on the right hand side of (23); taking integration on R_a by parts, we obtain that

$$\begin{aligned} R_a = & \frac{\alpha}{\sqrt{\omega}} [F(x_{k_0}, s) \cos(\omega s)]_0^t - \frac{\alpha}{\sqrt{\omega}} \int_0^t \cos(\omega s) \dot{F}(x_{k_0}, s) ds \\ = & \frac{\alpha}{\sqrt{\omega}} [F(x_{k_0}, s) \cos(\omega s)]_0^t \\ & - \frac{\alpha}{\sqrt{\omega}} \int_0^t \cos(\omega s) \frac{\partial F(x_{k_0}, s)}{\partial t} ds \\ & - \frac{\alpha}{\sqrt{\omega}} \int_0^t \cos(\omega s) \nabla_x F(x_{k_0}, s) \dot{x}_{k_0} ds \end{aligned}$$

Denote the 1st and 2nd terms in the right hand side of the above equation as R_1 and R_2 respectively. Iterating (4) to the first iteration, it follows that

$$\dot{x}_{k_0} = - \sum_{i=1}^{k_0} (1-\beta)^{k_0-i} \alpha F(x_i, t) \sqrt{\omega} \sin(\omega t) + \gamma_{k_0} \sqrt{\omega} \cos(\omega t) \quad (24)$$

Inserting (24) into R_a , it becomes

$$\begin{aligned} R_a = & R_1 + R_2 + R_3 - \alpha \gamma_{k_0} \int_0^t \cos^2(\omega s) \nabla_x F(x_{k_0}, s) ds \\ = & R_1 + R_2 + R_3 + R_4 - \frac{\alpha \gamma_{k_0}}{2} \int_0^t \nabla_x F(x_{k_0}, s) ds \end{aligned}$$

Here $R_3 \triangleq \frac{\alpha^2}{2} \sum_{i=1}^{k_0} (1-\beta)^{k_0-i} \int_0^t \sin(2\omega s) \nabla_x F(x_{k_0}, s) F(x_i, s) ds$ and $R_4 \triangleq -\frac{\alpha \gamma_{k_0}}{2} \int_0^t \cos(2\omega s) \nabla_x F(x_{k_0}, s) ds$. The second equality stems from the identity $\cos(2\omega s) = 2 \cos^2(\omega s) - 1$. Hence, $x_{k_0} - z_{k_0}$ becomes

$$x_{k_0} - z_{k_0} = \sum_{i=1}^5 R_i - \frac{\alpha \gamma_{k_0}}{2} \int_0^t [\nabla_x F(x_{k_0}, s) - \nabla_z F(z_{k_0}, s)] ds \quad (25)$$

where $R_5 = \sin(\omega t) / \sqrt{\omega}$. It should be noted that (25) holds universally for any t . Now we are going to show x_{k_0} will remain in S over $[0, L]$ by contradiction. Let $E = \text{dist}(\mathcal{K}, S)$. Assume that there is a time instant $t_E \in (0, L)$ such that $\|x_{k_0}(t) - z_{k_0}(t)\|_\infty < E$ for any $t \in [0, t_E)$ and $\|x_{k_0}(t_E) - z_{k_0}(t_E)\|_\infty = E$. It means that t_E is the first time when x_{k_0} leaves a tube with z_{k_0} as center and radius E . Since S is a compact set, for any $x \in S$, there always exist $\epsilon_1, \epsilon_2, \epsilon_3$ such that

$$\|F(x, s)\|_\infty \leq \epsilon_1, \|\nabla_x F(x, s)\|_\infty \leq \epsilon_2, \left\| \frac{\partial F(x, s)}{\partial t} \right\|_\infty \leq \epsilon_3$$

Note that these bounds are uniformly bounded in t . Thus, we have

$$\begin{aligned} \|R_1\|_\infty &= \left\| \frac{\alpha}{\sqrt{\omega}} [F(x_k, s) \cos(\omega s)]_0^t \right\|_\infty \leq \frac{2\alpha\epsilon_1}{\sqrt{\omega}} \\ \|R_2\|_\infty &= \left\| \frac{\alpha}{\sqrt{\omega}} \int_0^t \cos(\omega s) \frac{\partial F(x_k, s)}{\partial t} ds \right\|_\infty \leq \frac{\alpha\epsilon_3}{\omega^{3/2}} \\ \|R_3\|_\infty &= \left\| \frac{\alpha^2}{2} \sum_{i=1}^{k_0} (1-\beta)^{k_0-i} \int_0^t \sin(2\omega s) \nabla_x F(x_k, s) F(x_i, s) ds \right\|_\infty \\ &\leq \frac{\alpha^2 \epsilon_1 \epsilon_2}{2\omega} \sum_{i=1}^{k_0} (1-\beta)^{k_0-i} \\ &< \frac{\alpha^2 \epsilon_1 \epsilon_2}{2\beta\omega} \\ \|R_4\|_\infty &= \left\| \frac{\alpha \gamma_{k_0}}{2} \int_0^t \cos(2\omega s) \nabla_x F(x_{k_0}, s) ds \right\|_\infty \leq \frac{\alpha \gamma_{k_0} \epsilon_2}{4\omega} \\ &\leq \frac{\alpha \epsilon_2}{4\beta\omega} \\ \|R_5\|_\infty &= \left\| \frac{\sin(\omega s)}{\sqrt{\omega}} \right\|_\infty \leq \frac{1}{\sqrt{\omega}} \end{aligned}$$

So there must exist a positive number M such that $\sum_{i=1}^5 R_i \leq \frac{M}{\sqrt{\omega}}$ and M is independent of k .

Since $F(x, t)$ are twice continuous on $\mathcal{S} \times [0, t_E]$, there exist a positive number K such that $\|\nabla_x F(x_{k_0}, s) - \nabla_z F(z_{k_0}, s)\|_\infty \leq K \|x_{k_0} - z_{k_0}\|_\infty$ according to Lemma 3.2 (pp. 90, [25]). Thereafter, for $t \in [0, t_E]$

$$\|x_{k_0} - z_{k_0}\|_\infty \leq \frac{M}{\sqrt{\omega}} + \frac{\alpha K}{2\beta} \int_0^t \|x_{k_0} - z_{k_0}\|_\infty ds$$

From Gronwall-Bellman inequality (pp. 651, [25]), we have

$$\|x_{k_0} - z_{k_0}\|_\infty \leq \frac{M}{\sqrt{\omega}} e^{\frac{\alpha K}{2\beta} t}, t \in [0, t_E] \quad (26)$$

For any $\omega_0 \in (M^2 e^{\frac{\alpha K t_E}{\beta}} / E^2, \infty)$, where ω_0 is independent of k and t , $\|x_{k_0}(t_E) - z_{k_0}(t_E)\|_\infty < E$, which contradicts. Thus, x_{k_0} will remain in the tube of z_{k_0} . Since z_{k_0} stays in \mathcal{K} , x_{k_0} will stay in S . (26) suggests that $\|x_k - z_k\|_\lambda$ can be small enough if ω is large enough. This completes the proof. ■

APPENDIX D

PROOF OF THEOREM 4.5

Proof: From Lemma 4.3 and Theorem 4.4, we know that there exists a ω_0 for every $\omega \in (\omega_0, +\infty)$ such that we can ensure that $\|x_k - z_k\|_\lambda \leq D_1$ and $y_k \in S$ for any k .

By similar arguments in the proof of Theorem 4.2, from the definition of limit, for an arbitrarily chosen $\epsilon > 0$, selecting an $\epsilon' > 0$ satisfying $2\epsilon'/(1-\rho) < \epsilon$,

$$\frac{2\epsilon' D_2}{1-\rho} < \epsilon$$

there exists a k_ϵ such that $\|G_k - G_\infty\| < \epsilon'$ can be guaranteed as long as $k \geq k_\epsilon$.

From (22), \mathcal{Y} is a λ -norm ball with y_∞ as its center and D_y as its radius. Hence, we have

$$\text{dist}(y_k, \mathcal{Y}) = \max \{\|y_k - y_\infty\|_\lambda - D_y, 0\}$$

If $\text{dist}(y_k, \mathcal{Y}) = 0$, it means that $y_k \in \mathcal{Y}$.

For arbitrary $k \geq k_\epsilon$, we have the following from (10),(16).

$$\begin{aligned} & \|y_k - y_\infty\|_\lambda \\ &= \|G_k(y_{k-1}) - G_\infty(y_\infty) + T_k(x_{k-1} - z_{k-1})\|_\lambda \\ &\leq \|G_k(y_{k-1}) - G_k(y_\infty)\|_\lambda + \|G_k(y_\infty) - G_\infty(y_\infty)\|_\lambda \\ &\quad + \|T_k(x_{k-1} - z_{k-1}) - T_k(0)\|_\lambda \\ &\leq \rho \|y_{k-1} - y_\infty\|_\lambda + 2\epsilon' D_2 + \rho D_1 \end{aligned}$$

Iterating the above equation to k_ϵ , we have

$$\begin{aligned} \|y_k - y_\infty\|_\lambda &\leq \rho^{k-k_\epsilon} \|y_{k_\epsilon} - y_\infty\|_\lambda + \sum_{i=0}^{k-k_\epsilon-1} 2\rho^i \epsilon' D_2 \\ &\leq \rho^{k-k_\epsilon} \|y_{k_\epsilon} - y_\infty\|_\lambda + \frac{2\epsilon' D_2}{1-\rho} + \frac{\rho D_1}{1-\rho} \end{aligned}$$

It follows from Theorem 4.4 that $\|y_{k_\epsilon} - y_\infty\|_\lambda$ is bounded by $2D_2$. Therefore, it can be guaranteed that $\|y_k - y_\infty\|_\lambda \leq D_y + \epsilon$ for any k satisfying

$$k > k_\epsilon + \log_\rho \frac{1}{2D_2} \left(\epsilon - \frac{2\epsilon' D_2}{1-\rho} \right)$$

ϵ is chosen arbitrarily; thus, we can conclude that

$$\lim_{k \rightarrow \infty} \|y_k - y_\infty\|_\lambda \leq \frac{\rho D_1}{1-\rho}$$

It follows that

$$\lim_{k \rightarrow \infty} \text{dist}(y_k, \mathcal{Y}) = 0$$

That means y_k will finally be within \mathcal{Y} . This completes the proof. ■

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